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# Generalizations of Singular Optimal Control Theory†

Généralizations de la théorie de commande optimale singulière

Verallgemeinerung der Theorie der singulären Optimalwertregelung

Обобщения теории необычного оптимального управления

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*With appropriate transformations, linear regulator theory can be used to simplify substantially the determination of optimal controls for a class of singular problems.*

**Summary**—This paper considers a new approach to the optimization of the linear, possibly time-varying, system

$$\dot{x}=Fx+Gu \qquad |u_i|\leq 1$$

with respect to the performance index

$$V=\int_{t_0}^{t_1} x'Qx\mathrm{d}t.$$

The new approach applies standard regulator theory using appropriate transformations and thereby enables the problem to be solved more completely than has hitherto been possible. For example we consider the cases which arise when $Q$ is singular.

Particular attention is given to the limiting case as $t_1$ becomes infinite. For this case, conditions are presented for the asymptotic stability of the singular optimal trajectories.

Some stability results concerning the "bang-bang" solutions are also considered.

## 1. INTRODUCTION

A PROBLEM in optimal control which has been discussed for some years, but never fully solved, is the following: Given the completely controllable system

$$\dot{x}(t)=F(t)x(t)+G(t)u(t) \tag{1}$$

with the initial condition $x(t_0)=x_0$ and the constraints

$$|u_i|\leq 1,\ i=1,\ldots,m \tag{2}$$

where $u_i$ is the $i$th component of the $m$-vector $u$; then find $u$ such that the performance index

$$V\{u\}=\int_{t_0}^{t_1} x'Qx\mathrm{d}t \tag{3}$$

is minimized. It is normally assumed§ that $Q=Q'\geq 0$, and that $F(t)$, $G(t)$ and $Q(t)$ have bounded entries for all $t$.

A preliminary study, using the Pontryagin maximum principle, shows quite easily that the solution may have, in general, three modes:

(i) A "bang-bang" mode, where $|u_i|=1$ for all $i$. The value of $u_i$ may switch between $u_i=+1$ and $u_i=-1$.

(ii) A "singular" mode, where $|u_i|<1$ for all $i$. In this case the maximum principle provides very little information about the optimal $u$.

(iii) A "partially singular" mode, where $|u_i|<1$ for some but not all $i$.

Although the solution for the "bang-bang" case is an interesting problem in itself, we will be mainly concerned here with the singular case.

Several authors, notably WONHAM and JOHNSON [1] and BASS and WEBBER [2, 3] have considered the problem under the following assumptions:

(a) $u$ is a scalar, i.e. $m=1$;
(b) $Q$, $F$ and $G$ are constant matrices;
(c) $Q$ is positive definite,

and have shown that $u^*$, the optimal value of $u$, is of the form

$$u^*=k'x$$

on the "singular strip" $|k'x|\leq 1$, $k_1'x\equiv 0$, where $k$ and $k_1$ are constants. Here, the term "singular

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§ The notation $A>B(A\geqslant B)$ will henceforth be used to mean that the matrix $(A-B)$ is positive definite (non-negative definite).

strip" is used to denote that region of the state space in which the mode of operation is the "singular" mode. Outside the singular strip, the optimal control is apparently "bang-bang", although no definite results on this point are available. To the authors' knowledge, the theory of Refs. [1–3] cannot be extended to the case where the assumptions (a–c) do not hold.

More recently, ROHRER and SOBRAL [4] have developed results which do not require assumptions (a) and (b). On the other hand, the work of SIRISENA [5] does not require assemption (c). However, it is not at all clear how the work of [4] or [5] can be extended to include the more general case when assumptions (a), (b) and (c) are not required. Both Refs. [4] and [5] discuss the need to have a theory where the above assumptions are not made.

This paper presents a new approach to the singular control problem which is more general than those given previously. Using a transformation discussed in a companion note [6], the singular mode control problem is interpreted as a standard linear regulator problem in disguise. The method is applicable to multiple input, time-varying systems where the matrix $Q$ may be singular. That is, assumption (a), (b) and (c) are not required. In contrast to some of the earlier treatments the method does not require a special co-ordinate basis. One specific application is discussed in [7] where a standard regulator problem is solved with the added constraint that $|\dot{u}_i| \leq 1$ for all $i$.

The question of the stability of the optimal trajectories is also discussed. This is not a serious problem in the earlier treatments [1–4], since it turns out that when $Q > 0$, all singular optimal trajectories are asymptotically stable. However, when $Q$ is allowed to be singular there is a possibility of unstable solutions. The last section of the paper summarises sufficient conditions for asymptotic stability of the optimal solutions.

## 2. FINITE TIME RESULTS

*Method of solution*

In this section we consider the solution of the optimization problem (1–3) when the final time $t_1$ is finite. We note in passing that the final state $x(t_1)$ is unspecified; an indirect constraint could be made on this state by adding a term $x'(t_1)Ax(t_1)$ to the performance index, but as our interest will ultimately centre on the infinite time problem, we shall not pursue this line of argument further.

As a first step in solving the optimization problem we consider the minimization of the index (3) subject to the system equation (1) *but not subject to the inequality constraints* (2). Once this problem has been solved we will discard those solutions which violate the constraints on $u$.

In order to solve for the singular solutions of the above problem we define new variables $u_1$ and $x_1$ through

$$\dot{u}_1 = u \tag{4}$$

$$x_1 = x - Gu_1 \tag{5}$$

where the boundary condition for equation (4) will be specified later. Substituting the new variables into (1) and (3), we find that

$$\dot{x}_1 = Fx_1 + G_1 u_1 \tag{6}$$

and

$$V = \int_{t_0}^{t_1} (x_1' Q x_1 + 2x_1' S_1 u_1 + u_1' R_1 u_1) \mathrm{d}t \tag{7}$$

where

$$G_1 = FG - \dot{G}; \qquad S_1 = QG; \qquad R_1 = G'QG. \tag{8}$$

We now see that the transformations (4) and (5) have converted the singular minimization problem into the quadratic regulator problem as solved by KALMAN [8], at least for the case when $R_1$ is positive definite. For the case when $R_1 = G'QG$ is not positive definite, the index will obviously not have the standard form and further transformations are necessary. These cases will be considered later.

*The case $G'QG > 0$*

Without further comment, we shall restrict attention to the case when $R_1 = G'QG$ is positive definite. Note that when $Q$ is positive definite this condition is automatically satisfied if the inputs are independent or if there is one input, and even in the case when $Q$ is singular there are many possible $G$ matrices such that $R > 0$. Given this condition, it is simple to show, as indicated in Appendix I, that $(Q - S_1 R_1{}^{-1} S_1')$ is nonnegative definite, and hence the results of [8] may be applied directly. For the finite time case the optimal control $u_1{}^*$ is given by

$$u_1{}^* = K_1' x_1 \tag{9}$$

and the minimum index $V^*$ is given as

$$V^*(x_1(t_0), t_0) = x_1'(t_0) P(t_0, t_1) x_1(t_0) \tag{10}$$

where $K_1$ is given from

$$K_1' = -R_1^{-1}(G_1' P + S_1') \tag{11}$$

and $P(\cdot, t_1)$, where existence is guaranteed, is the solution of the Riccati differential equation

$$\begin{aligned} -\dot{P} &= P(F - G_1 R_1^{-1} S_1') + (F - G_1 R_1^{-1} S_1')' P \\ &\quad - P G_1 R_1^{-1} G_1' P + (Q - S_1 R^{-1} S_1'); \\ P(t_1, t_1) &= 0 \end{aligned} \tag{12}$$

Before interpreting the above results in terms of the desired solution of the finite time singular regulator problem, we will perform some manipulations using the various equations and definitions to yield some quite simple relationships between the various quantities hitherto defined. In particular, we shall establish, in the order listed, the relationships

$$PG \equiv 0,\ K_1'G \equiv -I, \text{ and } V^* = x'(t_0)P(t_0, t_1)x(t_0).$$

Postmultiplying both sides of the Riccati equation (12) by the matrix $G$ gives that

$$-\dot{P}G = P(F - G_1R_1^{-1}S_1')G + (F - G_1R_1^{-1}S_1')'PG - PG_1R_1^{-1}G_1PG + (Q - S_1R_1^{-1}S_1')G.$$

Applying the definitions (8) (namely, $R_1 = S_1'G$, $S_1 = QG$ and $G_1 = FG - \dot{G}$) gives immediately that

$$-\frac{\mathrm{d}}{\mathrm{d}t}(PG) = (F' - S_1R_1^{-1}G_1' - PG_1R_1^{-1}G_1')PG.$$

Now since $P(t_1, t_1)G(t_1) = 0$, the above differential equation in $(PG)$ has the solution

$$PG \equiv 0. \tag{13}$$

This is the first of the simple relationships.

Postmultiplying both sides of equation (11) by the matrix $G$ gives that

$$K_1'G = -R_1^{-1}(G_1'PG + S_1'G).$$

Applying the result $PG = 0$ and the definitions (8), in particular $R_1 = S_1'G$, yields the second simple result

$$K_1'G = -I, \text{ together with } \dot{K}_1'G + K_1'\dot{G} = 0. \tag{14}$$

For the case when the control $u_1$ is in fact the optimal control $u_1^*$ then equation (5) becomes

$$x_1 = x - Gu_1^* \tag{15}$$

and the minimum index $V^*$ given from equations (2)–(10) may be written as

$$V^*(x_1(t_0), t_0) = [x(t_0) - G(t_0)u_1^*(t_0)]'P(t_0, t_1)[x(t_0) - G(t_0)u_1^*(t_0)].$$

Since $PG = 0$ from (13) this becomes a function of only $x(t_0)$ and $t_0$ as follows

$$V^*(x(t_0), t_0) = x'(t_0)P(t_0, t_1)x(t_0). \tag{16}$$

We are now in a position to derive the equation of the singular strip; for when $u_1 = u_1^*$, we have from equation (5) that

$$K_1'x = K_1'x_1 + K_1'Gu_1^* = K_1'x_1 + K_1'GK_1'x_1$$

and using the result that $K_1'G = -I$, this reduces to

$$K_1'x = 0. \tag{17}$$

Essentially, equation (17) is the condition under which the transformation (4), (5) remains valid; or, as we shall see, a sufficient condition for the optimal control $u^*$ to be a "singular" control. Experience indicates that (17) is also a necessary condition for a singular optimal control, but so far this remains unproven in the general case.

Still considering the case when the control $u_1$ is optimal, that is when $u_1 = u_1^* = K_1'x_1$, we denote the control $u$ corresponding to this as $u^*$ without claiming for the moment that $u^*$ is optimal. We have from (4) and (9) that

$$u^* = \dot{u}_1^* = \dot{\overline{(K_1'x_1)}} = \dot{K}_1'x_1 + K_1'\dot{x}_1.$$

From the original system equation and the constraint equation (15), there follows

$$\begin{aligned} u^* &= \dot{K}_1'(x - Gu_1^*) + K_1'[F(x - Gu_1^*) + (FG - \dot{G})u_1^*] \\ &= \dot{K}_1'x + K_1' \ x - (\dot{K}_1'G + K_1'\dot{G})u_1^* \\ &= (K_1'F + \dot{K}_1')x \end{aligned}$$

where the final equality follows by use of (14). For convenience, we make the definition

$$K = F'K_1 + \dot{K}_1$$

so that

$$u^* = K'x. \tag{18}$$

*Optimality of the solutions*

The proof that the control $u^* = K'x$ is optimal is immediate, since a simple manipulation of equation (12) yields that

$$\int_{t_0}^{t_1} x'(t)Q(t)x(t)\mathrm{d}t = x'(t_0)P(t_0, t_1)x(t_0) + \int_{t_0}^{t_1} \{K_1'(t)x(t)\}'R_1\{K_1'(t)x(t)\}\mathrm{d}t$$

By assumption, $K_1'(t_0)x(t_0) = 0$. Thus, since $R_1$ is positive definite, any singular optimal trajectory $x^*(t)$ must satisfy $K_1'(t)x^*(t) = 0$ for all $t$ such that

$t_0 \leq t < t_1$—provided, of course, that this is possible. It is clear that $u^*$ as defined above is the unique control such that this condition is satisfied, therefore $u^*$ is the optimal control.

Of course, by simply rejecting those solutions of the singular minimization problem above which violate the constraints (2) at any time $t \geq t_0$, solutions that remain are the solutions to the singular optimal regulator problem posed at the beginning of this section where we included the control magnitude constraint. That is, for the input to operate in the singular mode rather than the bang-bang mode the constraints

$$|(K'x)_i| < 1 \quad \text{and} \quad (K_1'x)_i = 0 \tag{19}$$

must hold for all $i$ and all $t_0 \leq t \leq t_1$. Given these conditions, the optimal control is given by equation (18).

When the conditions (19) do not hold, the optimal control is in general bang-bang, and in fact $u^* = \text{sgn}(K_1'x)$ in some vicinity of the singular strip —see [3] for a detailed examination of this case.

*Further singular solutions: $G'QG = 0$*

We now consider the case where $R_1 = G'QG$ may be singular. We shall show in this case that the optimal control on the singular strip is still given by a linear feedback law, but that the dimensionality of the singular strip is reduced.

Consider first the case where $R_1$ is identically zero. Clearly, since $Q \geq 0$, $S_1$ is also zero, so that the transformed problem (6), (7) is once more singular. In this case we may use a new transformation

$$\dot{u}_2 = u_1$$
$$x_2 = x_1 - G_1 u_2$$

where $u_1$, $x_1$ and $G_1$ are defined as before. Then

$$\dot{x}_2 = Fx_2 + G_2 u_2$$

and

$$V = \int_{t_0}^{t_1} (x_2'Qx_2 + 2x_2'S_2u_2 + u_2'R_2u_2)\text{d}t$$

where

$$G_2 = FG_1 - \dot{G}_1\,; \qquad S_2 = QG_1\,; \qquad R_2 = G_1'QG_1\,.$$

If $R_2$ is nonsingular, the optimal control may be derived as before. If not, further transformations may be used.

Suppose in general that $j$ transformations are necessary before a non-zero $R_j$ is found, and suppose for the moment that $R_j$ is nonsingular.† One then has a series of parameters defined by

$$\left.\begin{aligned} G_i &= FG_{i-1} - \dot{G}_{i-1} \\ S_i &= QG_{i-1} \\ R_i &= G_{i-1}'QG_{i-1} \end{aligned}\right\} \qquad i = 1, \ldots, j$$

where $G_0$ is identified with $G$. At the $j$th step, then, the system becomes

$$\dot{x}_j = Fx_j + G_j u_j$$

with performance index

$$V = \int_{t_0}^{t_1} (x_j'Qx_j + 2x_j'S_ju_j + u_j'R_ju_j)\text{d}t\,.$$

In the same way as before, we obtain

$$u_j^* = K_j'x_j\,; \qquad V^* = x_j'Px_j$$

where

$$K_j = -(PG_j + S_j)R_j^{-1}$$

and $P(\cdot, t_1)$ is the solution of

$$\begin{aligned} -\dot{P} &= P(F - G_jR_j^{-1}S_j') + (F - G_jR_j^{-1}S_j')'P \\ &\quad - PG_jR_j^{-1}G_j'P - S_jR_j^{-1}S_j' + Q\,; \\ &P(t_1, t_1) = 0. \end{aligned} \tag{20}$$

The reverse transformation, from $u_j^*$ to $u^*$, proceeds in much the same way as before—detailed calculations and proofs are given in Appendix 2. The final result is that

$$u^* = K'x \tag{21}$$

where $K$ is computed recursively from the relationship

$$K_{i-1} = F'K_i + \dot{K}_i$$

in which $K_0$ is to be identified with $K$. Also

$$V^*(x(t), t) = x'(t)P(t, t_1)x(t) \tag{22}$$

and the singular strip lies on the intersection of the hyperplanes

$$K_i'x = 0, \qquad i = 1, \ldots, j. \tag{23}$$

Clearly, the transformations are only valid if we restrict the singular strip to that subset of (23) for which $|K'x| < 1$ for all $t_0 \leq t \leq t_1$. It seems plausible,

† It is possible that no singular solutions exist, in which case a non-zero $R_j$ will never be found in the range $j < n$ ($n$ = order of the system).

although no proof is available, that this inequality defines the largest subset of (23) for which every component of the optimal control is singular.

*Singular solutions when $R_j$ is singular*

For completeness we now mention the case—not encountered in single input systems—for which the above transformations yield a *singular* but non-zero $R_j$. Without loss of generality, we may assume $R_j$ to be of the form

$$R_j=\begin{bmatrix} R_a & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, with appropriate partitioning of the various coefficient matrices, we find that

$$\dot{x}_j=Fx_j+G_a u_a+G_b u_b \tag{24}$$

and

$$V=\int_{t_0}^{t_1}(x_j'Qx_j+2x_j'S_a u_a+u_a'R_a u_a)\mathrm{d}t \tag{25}$$

where we define

$$u=\begin{bmatrix} u_a \\ u_b \end{bmatrix}.$$

It may be seen that the performance index is singular in the $u_b$ components, but not in the $u_a$ components of the control vector. With this in mind, the transformation used earlier is now applied to $u_b$ only—that is, we set

$$\dot{u}_{b+1}=u_b$$
$$x_{j+1}=x_j-G_b u_{b+1}.$$

Making the appropriate substitutions in (24) and (25), it is found that

$$\dot{x}_{j+1}=Fx_{j+1}+G_{j+1}u_{j+1}$$

and

$$V=\int_{t_0}^{t_1}(x_{j+1}'Qx_{j+1}+2x_{j+1}'S_{j+1}u_{j+1} + u_{j+1}'R_{j+1}u_{j+1})\mathrm{d}t$$

where

$$u_{j+1}=\begin{bmatrix} u_a \\ u_{b+1} \end{bmatrix}$$

and the other matrices are readily calculated. The subsequent calculations are tedious, but it should be clear that, if $R_{j+1}$ is nonsingular, $u_{j+1}$ may be found via a Riccati equation, and thence $u_j$ is readily found. From this point, the results precisely parallel those given in equations (20–23). There is, however, one extra equation to be added to (23), namely

$$K_{b+1}'x=0$$

where $K_{b+1}$ is the gain matrix associated with $u_{b+1}$. This is obviously the condition for the "$b$" components of the input to be in their singular mode.

The only remaining possibility is that $R_{j+1}$ is also singular, in which case the procedure is to apply a further transformation.

## 3. INFINITE TIME RESULTS

The preceding results may all be extended to the limiting case, as the final time $t_1$ approaches infinity, provided only that the limit

$$\bar{P}(t)=\lim_{t_1\to\infty}P(t,t_1)$$

can be shown to exist. Since it is known, from elementary Riccati equation theory, that

$$x'(t)P(t,t_1)x(t)$$

increases monotonically as $t_1$ increases, because $Q-S_jR_j^{-1}S_j'\geq 0$, it is sufficient to prove that $x'(t)P(t,t)x(t)$ has an upper bound for an arbitrary $x(t)$ which is independent of $t_1$. This may be shown as follows.

Assume that the pair $[F, G]$ is completely controllable. Then for a given but arbitrary $x(t)$, it is straightforward to exhibit a control such that

$$V\{u\}=\int_t^{t_1}x'Qx\,\mathrm{d}t$$

is bounded above, independently of $t_1$. But it was shown earlier that, for $R_1>0$.

$$\int_t^{t_1}x'Qx\mathrm{d}t=x'(t)P(t,t_1)x(t)+\int_t^{t_1}(K_1'x)'R_1(K_1'x)\mathrm{d}t$$

and a similar result holds when $R_1$ is singular. The required upper bound therefore exists, so that $\bar{P}(t)$ always exists when $[F, G]$ is completely controllable.

*Stability results*

An important motivation for constructing optimal systems is that such systems tend to have desirable engineering properties such as a high degree of stability, tolerance of nonlinearities without appreciable loss of performance, and a low sensitivity to plant parameter variations. In particular, the optimal performance index may often be shown to be a Lyapunov function. References [9] and [10] show that these properties are, in fact, present in singular optimal systems of the type considered here. The main results are as follows:

1. Assume that the pair $[F+GK', D]$ is uniformly completely controllable for some $D$ such that $DD'=Q$. Then the optimal performance index is a Lyapunov function for all singular trajectories, which are therefore stable trajectories.

2. All optimal "bang-bang" trajectories sufficiently near the singular strip ultimately reach the singular strip, so that the above assumption guarantees local asymptotic stability near $x=0$.

3. With the additional assumption that the free system $\dot{x}=Fx$ is exponentially asymptotically stable, the optimal system is globally asymptotically stable.

4. The above stability results continue to hold if unintentional nonlinearities, of a fairly general type, are introduced at the plant input.

It is also possible to extend the results of [11] in a straightforward way to ensure that the optimal singular trajectory $x^*$ decays at least as fast as $e^{-\alpha t}$, where $\alpha$ is any specified positive number.

## 4. CONCLUSIONS

Singular solutions are found to exist on hyperplanes of dimension lower than the order of the system for any nonnegative definite $Q$. In all cases the present procedure gives the equations of the singular hyperplanes, and the optimal singular control in the form of a linear feedback law. For time-invariant single-input systems and special choices of the matrix $Q$, these results reduce to the results of [1–5].

A minor computational difficulty arises in attempting to solve the infinite time problem, since the associated steady state Riccati equation may have more than one nonnegative definite solution. Thus, although any of the standard solution methods [12] may still be used, there is no guarantee that the solution found will in fact correspond to the limiting solution of the finite-time problem. In practice, the authors have found that it is preferable to solve for the finite-time solutions first, and to obtain the steady state solution by a limiting method.

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## APPENDIX 1

*Proof that* $Q-S_1R_1^{-1}S_1'\geq 0$

Using the known expression for $S_1$, we have that

$$Q-S_1R_1^{-1}S_1'=Q-QGR_1^{-1}G'Q.$$

Since $Q$ is symmetric and nonnegative definite, we can find an $H$ such that $Q=H'H$. Then

$$\begin{aligned}Q-S_1R_1^{-1}S_1'&=H'H-H'HGR_1^{-1}G'H'H\\&=H'[I-HGR_1^{-1}G'H']H.\end{aligned}$$

Now choose a matrix $V$ such that

$$HG=VR_1^{\frac{1}{2}}$$

Since $R_1$ is nonsingular, such a choice may always be made (and is unique). We have then that

$$G'H'HG=R_1^{\frac{1}{2}}V'VR_1^{\frac{1}{2}}$$

or

$$R_1=R_1^{\frac{1}{2}}V'VR_1^{\frac{1}{2}}$$

which implies that

$$V'V=I.$$

Substituting for $HG$ in the above expression,

$$Q-S_1R_1^{-1}S_1'=H'[I-VV']H.$$

Now since

$$V'V=I$$

$$VV'VV'=VV'$$

that is,

$$(VV')^2=VV'$$

and hence $\lambda^2=\lambda$ where $\lambda$ is any eigenvalue of $VV'$. Obviously, then, all eigenvalues of $VV'$ are 0 or 1, and hence all eigenvalues of $(I-VV')$ are 1 or 0. Since $(I-VV')$ is a symmetric matrix, it follows that

$$(I-VV')\geq 0$$

and therefore

$$Q-S_1R_1^{-1}S_1'=H'[I-VV']H\geq 0$$

as required.

*Note.* The proof is equally valid, with obvious modifications, for the case where $j$ transformations are necessary to yield a non-zero $R_j$, the result then being

$$Q-S_jR_j^{-1}S_j'\geq 0.$$

## APPENDIX 2

*Derivation of equations* (21–23)

Our starting point is the Riccati equation

$$\begin{aligned} -\dot{P}=P(F-G_jR_j^{-1}S_j')+(F-G_jR_j^{-1}S_j')'P \\ -PG_jR_j^{-1}G_j'P-S_jR_j^{-1}S_j' \\ +Q;\ \ P(t_1,t_1)=0 \end{aligned} \qquad (20)$$

and the definition

$$K_j=-(PG_j+S_j)R_j^{-1}.$$

In the same manner as before, it is immediate that

$$PG_{j-1}\equiv 0,\quad \text{and}\quad K_j'G_{j-1}\equiv -I. \qquad (55)$$

Now, noting that $QG_i=0$ for all $i<j$, equations (20) and (55) yield, by induction,

$$PG_i\equiv 0 \quad \text{for all} \quad i<j. \qquad (56)$$

It follows immediately from (55) and (56) that

$$K_j'G_i\equiv 0 \quad \text{for all} \quad i<j-1.$$

If we define a set of vectors $K_1$ by

$$K_{i-1}=\dot{K}_i+F'K_i \qquad (57)$$

(i.e. the $K_i$ are computed backwards from $K_j$), then simple induction using (56) and (57) gives the indentities

$$\text{and}\quad \left.\begin{aligned} K_i'G_{i-1}=-I \\ K_i'G_{i-k}=0,\ k>1 \end{aligned}\right\}\text{for all } i\leq j. \qquad (58)$$

We are now in a position to compute $u^*$ from $u_j^*$. Recalling that $u_{j-1}^*=\dot{u}_j^*$, we obtain

$$u_{j-1}^*=\dot{K}_j'x_j+K_j'\dot{x}_j$$

which readily reduces to

$$u_{j-1}^*=K_{j-1}'x_{j-1}$$

by use of the above equations and the transformation $x_j=x_{j-1}-G_{j-1}u_j$. Successive use of this procedure gives $u_i^*=K_i'x_i$ for all $i\leq j$, and in particular

$$u^*=K'x. \qquad (21)$$

The optimal performance index

$$\begin{aligned} V^*&=x_j'Px_j \\ &=(x_{j-1}-G_{j-1}u_j)'P(x_{j-1}-G_{j-1}u_j) \end{aligned}$$

or, since $PG_{j-1}\equiv 0$,

$$V^*=x_{j-1}'Px_{j-1}$$

and by (56) this reduction can be continued, so that finally

$$V^*=x'Px. \qquad (22)$$

From the basic transformation $x_i=x_{i-1}-G_{i-1}u_i$, we can prove as before that $K_i'x_{i-1}\ldots 0$ for all $i$ in the range $1\leq i\leq j$. From this identity and equation (58), it follows by induction that

$$K_i'x\equiv 0,\qquad i=1,\ldots,j. \qquad (23)$$

This completes the derivation.

**Résumé**—Le présent article considère une nouvelle approche à l'optimalisation du système linéaire, eventuellement variable dans le temps,

$$x=Fx+Gu \qquad |u_i|\leqslant 1$$

par rapport à l'indice de performance

$$V=\int_{t_0}^{t_1}x'Qx\mathrm{d}t.$$

La nouvelle approche applique la théorie des régulateurs habituelle, en utilisant des transformations appropriées, et permet ainsi une solution plus complète du problème que n'a été possible jusqu'ici. Par exemple, l'article considère les cas qui ont lieu lorsque $Q$ est singulier.

Une attention particulière est reservée au cas-limite lorsque $t_1$ devient infini. Pour ce cas, l'article présente des conditions pour la stabilité asymptotique des trajectoires optimales singulières.

L'article considère également certains résultats se rapportant aux solutions par plus, moins ou zéro.

**Zusammenfassung**—Betrachtet wird ein neuer Zugang zur Optimierung des linearen, möglicherweise zeitvariablen Systems

$$x = Fx + Gu \qquad |u_i| \leqslant 1$$

hinsichtlich des Index der Arbeitsweise

$$V = \int_{t_0}^{t_1} x'Qx\mathrm{d}t\,.$$

Der neue Zugang verwendet die übliche Reglertheorie unter Benutzung geeigneter Transformationen und erlaubt deshalb eine vollständigere Lösung des Problems, als es bisher möglich war. Als Beispiel werden die sich ergebenden Fälle betrachtet, wenn $Q$ singulär ist.

Spezielle Aufmerksamkeit wird dem Grenzfall geschenkt, wenn $t_1$ unendlich wird. Für diesen Fall werden Bedingunge für die asymptotische Stabilität der singulären optimalen Trajektorien angegeben.

Einige Stabilitätsergebnisse, die "bang-bang" Lösungen betreffend, werden betrachtet.

**Резюме**—Настоящая статья рассматривает новый подход к оптимизации линейной, возможно переменной по времени, системы

$$x = Fx + Gu \qquad |u_i| \leqslant 1$$

по отношению к показателю работы

$$V = \int_{t_0}^{t_1} x'Qx\mathrm{d}t\,.$$

Новый подход применяет обычную теорию регуляторов, используя подходящие превращения, и позволяет таким образом более полное решение проблемы чем это было возможно до сих пор. Например, статья рассматривает случаи которые имеют место когда $Q$ становится необычным.

Особое внимание уделено пределбному случаю когда $t_1$ становится бесконечным. Для этого случая, статья предлагает условия асимптотической устойчивости для необычных оптимальных траекторий.

Статья также рассматривает некоторые результаты относящиеся к трехпозиционным решениям типа "плюс, минус, ноль".